DENSITY OF INTEGER SOLUTIONS TO DIAGONAL QUADRATIC FORMS

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Abstract. Let Q be a non-singular diagonal quadratic form in at least four variables. We provide upper bounds for the number of integer solutions to the equation $Q = 0$, which lie in a box with sides of length $2B$, as $B \to \infty$. The estimates obtained are completely uniform in the coefficients of the form, and become sharper as they grow larger in modulus.

1. Introduction

Let $n \geqslant 3$ and let $Q \in \mathbb{Z}[x_1, \ldots, x_n]$ be a non-singular indefinite quadratic form. Given an arbitrary bounded subset $\mathcal{R}$ of $\mathbb{R}^n$, it is natural to investigate the number of zeros $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ of the equation $Q(\mathbf{x}) = 0$ that are confined to the region $B\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n : B^{-1}\mathbf{x} \in \mathcal{R}\}$, as $B \to \infty$. In the present paper we will focus upon the case of integer solutions to diagonal quadratic forms, that lie in the box corresponding to taking $\mathcal{R}$ the unit hypercube in $\mathbb{R}^n$. Suppose once and for all that

$$Q(\mathbf{x}) = A_1 x_1^2 + \cdots + A_n x_n^2, \tag{1.1}$$

for non-zero integers $A_1, \ldots, A_n$ not all of the same sign, and write $\Delta_Q = A_1 \cdots A_n$ for the discriminant of Q. Our goal is therefore to understand the asymptotic behaviour of the counting function

$$N(Q; B) = \#\big\{\mathbf{x} \in \mathbb{Z}^n : \ Q(\mathbf{x}) = 0, \ |\mathbf{x}| \leqslant B\big\},$$

where $|\mathbf{x}| = \max_{1 \leqslant i \leqslant n} |x_i|$ denotes the usual norm on $\mathbb{R}^n$. Using Möbius inversion it is then possible to extract information about the corresponding counting function in which one is only interested in counting primitive vectors. This amounts to counting rational points of bounded height on the quadric hypersurface $Q = 0$ in $\mathbb{P}^{n-1}$. It will suffice to restrict our attention to primitive quadratic forms throughout our work, in the sense that $A_1, \ldots, A_n$ have greatest common divisor 1.

It should come as no surprise that the quantity $N(Q; B)$ has received substantial attention over the years, to the extent that many authors have established asymptotic formulae for quantities very similar to $N(Q; B)$. Let us define the more general counting function

$$N_w(Q; B) = \sum w(B^{-1}\mathbf{x}),$$

for suitable bounded weight functions $w : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ of compact support, where the summation is taken over all $\mathbf{x} \in \mathbb{Z}^n$ such that $Q(\mathbf{x}) = 0$. In particular we clearly have $N(Q; B) = N_{w^*}(Q; B)$, where $w^* = \chi_{[-1,1]^n}$ is the characteristic function of the unit hypercube in $\mathbb{R}^n$. Several methods have been developed to study $N_w(Q; B)$ for appropriate weight functions w, and we proceed to discuss what is known. Under

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suitable assumptions about w, Malyšev [8] has established an asymptotic formula for $N_w(Q;B)$ when $n \geqslant 5$, and Siegel [9] has done the same when $n=4$ and the discriminant Δ_Q is a square. One of the most impressive results in this direction, however, is due to Heath-Brown [6], who has established an asymptotic formula for $N_w(Q;B)$ when $n \geqslant 3$ and w belongs to a rather general class of infinitely differentiable weight functions. Heath-Brown's approach is based upon the Hardy–Littlewood circle method, and the outcome of his investigation is the existence of a non-negative constant $c(w;Q)$ such that

$$N_w(Q;B) = c(w;Q)B^{n-2}(\log B)^{b_n}\big(1+o(1)\big), \tag{1.2}$$

as $B \to \infty$. Here $b_n = 1$ if $n=3$, or if $n=4$ and Δ_Q is a square, and $b_n = 0$ otherwise. The constant $c(w;Q)$ may be interpreted as a product of local densities. All of these estimates for $N_w(Q;B)$ share the common feature that they depend intimately upon the coefficients of the quadratic form under consideration.

The central theme of this paper is the finer question of whether it is possible to provide estimates for the counting function $N(Q;B)$, for suitable choices of Q, in which the dependence upon the coefficients of Q is made completely explicit. Let

$$m(Q) = \min_{1\leqslant i\leqslant n} |A_i|, \quad \|Q\| = \max_{1\leqslant i\leqslant n} |A_i| \tag{1.3}$$

denote the minimum and height of Q, respectively. For small values of n the geometry of numbers is particularly effective for this sort of problem. Thus when $n=3$ it follows from the author's joint work with Heath-Brown [3, Corollary 2] that

$$N(Q;B) \ll \Big(\frac{h_Q^{1/2}B}{|\Delta_Q|^{1/3}}+1\Big)d(|\Delta_Q|), \tag{1.4}$$

where h_Q is the greatest common divisor of A_1A_2, A_1A_3 and A_2A_3, and d denotes the divisor function. When $n=4$ work of the author [1, Theorem 3] establishes that

$$N(Q;B) \ll_\varepsilon \frac{B^{2+\varepsilon}}{m(Q)^{1/3}|\Delta_Q|^{1/6}} + B^{3/2+\varepsilon}, \tag{1.5}$$

for any $\varepsilon > 0$, under the assumption that Δ_Q is square-free. With more care, $B^{2+\varepsilon}$ can be replaced by $B^2|\Delta_Q|^\varepsilon$ in this estimate. Both (1.4) and (1.5) have the obvious feature of becoming sharper as the discriminant of the form grows larger. When $n \geqslant 5$ the best uniform estimate available is the estimate

$$N(Q;B) \ll_{\varepsilon,n} B^{n-2+\varepsilon}, \tag{1.6}$$

that is due to Heath-Brown [7, Theorem 2]. Our main result consists of an estimate for $N(Q;B)$ that bridges (1.5) and (1.6) for arbitrary $n \geqslant 4$.

Theorem 1. *Let $n \geqslant 4$ and assume that Δ_Q is not a square when $n=4$. Then for any $\varepsilon > 0$ and any $B \geqslant 1$, we have*

$$N(Q;B) \ll_{\varepsilon,n} \Big(\frac{B^{n-2}}{m(Q)^{1/2}\|Q\|^{1/2}} + \frac{\|Q\|^{2n+3}}{m(Q)^{3n/4+3}|\Delta_Q|^{1/2}}B^{(n-1+\delta_n)/2+\varepsilon}\Big)|\Delta_Q|^\varepsilon,$$

where

$$\delta_n = \begin{cases} 1, & \text{if } n \text{ is even and } n \geqslant 5,\\ 0, & \text{if } n \text{ is odd or } n=4. \end{cases} \tag{1.7}$$

In view of the upper bounds $1 \leqslant m(Q)$ and $m(Q)^{-1}|\Delta_Q| \leqslant \|Q\|^{n-1}$, it is clear that we can always take $m(Q)^{1/2}\|Q\|^{1/2} \geqslant |\Delta_Q|^{1/(2(n-1))}$ in Theorem 1. For a typical diagonal quadratic form one expects the coefficients to have equal order of magnitude $|\Delta_Q|^{1/n}$, so that there exist positive constants $c_1 \geqslant c_2$, depending only on n, such that

$$c_1|\Delta_Q|^{1/n} \geqslant \|Q\| \geqslant m(Q) \geqslant c_2|\Delta_Q|^{1/n}. \tag{1.8}$$

The following result is a trivial consequence of Theorem 1.

Corollary. *Let $n \geqslant 4$ and assume that Δ_Q is not a square when $n = 4$. Suppose that* (1.8) *holds for appropriate constants c_1, c_2. Then for any $\varepsilon > 0$ and any $B \geqslant 1$, we have*

$$N(Q;B) \ll_{\varepsilon,n} \Big(\frac{B^{n-2}}{|\Delta_Q|^{1/n}} + |\Delta_Q|^{3/4}B^{(n-1+\delta_n)/2+\varepsilon}\Big)|\Delta_Q|^{\varepsilon},$$

where δ_n is given by (1.7).

A standard probabilistic argument suggests that $N(Q;B)$ should have order of magnitude $|\Delta|^{-1/n}B^{n-2}$, at least on average. Our bounds are clearly consistent with this heuristic. A brief discussion of certain lower bounds for $N(Q;B)$, in the case $n = 4$, can be found in the author's earlier work upon this problem [1, §4].

It is somewhat annoying that the term $m(Q)$ should appear at all in the statement of Theorem 1. We can obtain estimates independent of $m(Q)$ by considering an alternative counting function. Given a parameter $X \geqslant 1$, let

$$M(Q;X) = \#\Big\{\mathbf{x} \in \mathbb{Z}^n : \ Q(\mathbf{x}) = 0, \ \max_{1\leqslant i\leqslant n} |A_i x_i^2| \leqslant X\Big\}.$$

We will deduce the following result rather easily from our proof of Theorem 1.

Theorem 2. *Let $n \geqslant 5$. Then for any $\varepsilon > 0$ and any $X \geqslant 1$, we have*

$$M(Q;X) \ll_{\varepsilon,n} \Big(\frac{X^{(n-2)/2}}{|\Delta_Q|^{1/2}} + \|Q\|^{n/2+\varepsilon}X^{(n-1+\delta_n)/4+\varepsilon}\Big)|\Delta_Q|^{\varepsilon},$$

where δ_n is given by (1.7).

It would not be hard to extend Theorem 2 to cover the case in which $n = 4$ and Δ_Q is not a square. Our approach to estimating $N(Q;B)$ and $M(Q;X)$ is based on Heath-Brown's new version of the Hardy–Littlewood circle method [6] that was used to establish (1.2). Whereas the classical form of the circle method (as described by Davenport [4], for example) is based on the equality

$$\int_0^1 e^{2\pi i\alpha n}\mathrm{d}\alpha = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z}\setminus\{0\}, \end{cases}$$

Heath-Brown works with a more sophisticated expression for this indicator function. The other main difference is the use of Poisson summation to introduce a family of complete exponential sums, rather than using the major and minor arc distinction that appears in the classical circle method.

The overall plan will be to establish a version of the asymptotic formula (1.2), for a suitable weight function $w : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, in which the error terms dependence on Q is made completely explicit. Once coupled with a uniform upper bound for the constant $c(w;Q)$, this will suffice for the proof of Theorem 1. It is worth highlighting that the classical form of the circle method could easily be used to establish a result of the type in Theorem 1 when $n \geqslant 5$. However, a double Kloosterman refinement

is needed to treat the case $n = 4$. Heath-Brown's approach already incorporates a single Kloosterman refinement when $n \geqslant 5$, which in itself yields a sharper error term. Moreover, the double Kloosterman refinement needed to handle the case $n = 4$ can be carried out with little extra trouble. There are a number of extra technical difficulties that need to be dealt with before Heath-Brown's method can be implemented, however. The most substantial of these involves pinning down the exact dependence of his estimates for certain exponential integrals upon the quadratic forms under consideration.

Theorem 1 can be extended in a number of obvious directions. In addition to covering the case in which $n = 4$ and the discriminant Δ_Q is a square, it is possible to handle non-diagonal indefinite quadratic forms. We have decided to pursue neither of these refinements here, however, choosing instead to focus upon the simplest situation for which we can provide the strongest results.

We end this section by introducing some of the basic conventions and notations that we will follow throughout this work. As is common practice, we will allow the small positive constant ε to take different values at different points of the argument. We will often arrive at estimates involving arbitrary parameters M, N. These will typically be non-negative or positive, but will always take integer values. Given any vector $\mathbf{z} \in \mathbb{R}^n$ we write $\int f(\mathbf{z})\mathrm{d}\mathbf{z}$ for the n-fold repeated integral of $f(\mathbf{z})$ over $\mathbb{R}^n$. Given $q \in \mathbb{N}$, a sum with a condition of the form $\mathbf{b} \,(\mathrm{mod}\ q)$ will mean a sum taken over $\mathbf{b} \in \mathbb{Z}^n$ such that the components of $\mathbf{b}$ run from 0 to $q - 1$. Finally, for any $\alpha \in \mathbb{R}$ we will write $e(\alpha) = e^{2\pi i\alpha}$ and $e_q(\alpha) = e^{2\pi i\alpha/q}$.

2. Preliminaries

In this section we bring together the principal ingredients in the proof of Theorems 1 and 2. As indicated above, the main idea is to establish a uniform version of (1.2), for a suitable weight function. Before introducing the weight that we will work with, we first elaborate upon the nature of the constant $c(w; Q)$ that appears in Heath-Brown's estimate. As is well-known to experts, we have $c(w; Q) = \sigma_\infty(w; Q)\mathfrak{S}(Q)$, where $\sigma_\infty(w; Q)$ corresponds to the singular integral, and $\mathfrak{S}(Q)$ is the singular series. Define the p-adic density of solutions to be

$$\sigma_p = \lim_{k\to\infty} p^{-k(n-1)}\#\{\mathbf{x} \ (\mathrm{mod}\ p^k) : Q(\mathbf{x}) \equiv 0 \ (\mathrm{mod}\ p^k)\}, \tag{2.1}$$

for any prime p. When these limits exist, the singular series is given by

$$\mathfrak{S}(Q) = \prod_p \sigma_p. \tag{2.2}$$

We will see shortly that $\mathfrak{S}(Q)$ is convergent for the forms considered here.

Consider the function $w_0 : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, given by

$$w_0(x) = \begin{cases} e^{-(1-x^2)^{-1}}, & |x| < 1, \\ 0, & |x| \geqslant 1. \end{cases} \tag{2.3}$$

Then w_0 is infinitely differentiable with compact support $[-1, 1]$. Let

$$c_0 = \int_{-\infty}^{\infty} w_0(x)\mathrm{d}x, \tag{2.4}$$

and define the function

$$\omega_\epsilon(x) = c_0^{-1}\epsilon^{-1}\int_{-\infty}^{x-\epsilon} w_0(\epsilon^{-1}y)\mathrm{d}y,$$

for given $\epsilon > 0$. It is easy to see that ω_ϵ takes values in $[0,1]$ and is infinitely differentiable, with compact support $[0,2\epsilon]$. In our work we will make use of the non-negative weight function

$$w^\dagger(\mathbf{x}) = w_0(x_1-2)\prod_{i\neq 1}\omega_{\frac{1}{2}}\Big(1-\frac{x_i}{x_1}\Big), \tag{2.5}$$

on $\mathbb{R}^n$. It is clear that $w^\dagger(\mathbf{x})$ is zero unless $1 \leqslant x_1 \leqslant 3$ and $0 \leqslant x_i \leqslant x_1$ for $i \geqslant 2$. In particular $w^\dagger$ is supported in the compact region $[1,3]\times[0,3]^{n-1}$.

We are now ready to record our asymptotic formulae for the weighted counting function $N_{w^\dagger}(Q;B)$. Let Q be the primitive quadratic form (1.1), and recall the definitions (1.3) of the minimum and height of Q. We may and will assume that the coefficient A_1 is positive, throughout our work. The following result will be used to handle the case $n \geqslant 5$ in Theorem 1.

Proposition 1. *Let $n \geqslant 5$. Then there exists a non-negative constant $\sigma_\infty(Q)$ such that*

$$N_{w^\dagger}(Q;B) = \sigma_\infty(Q)\mathfrak{S}(Q)B^{n-2} + O_{\varepsilon,n}\Big(\frac{\|Q\|^{2n+3+\varepsilon}}{A_1^{3n/4+3}|\Delta_Q|^{1/2}}B^{(n-1+\delta_n)/2+\varepsilon}\Big),$$

where δ_n is given by (1.7)*, $\mathfrak{S}(Q)$ is given by* (2.2)*, and*

$$\sigma_\infty(Q) \ll_n A_1^{-1/2}\|Q\|^{-1/2}. \tag{2.6}$$

Turning to the case $n=4$, for which we must assume that the discriminant is not a square, we have the the following result.

Proposition 2. *Let $n=4$ and assume that Δ_Q is not a square. Then there exists a non-negative constant $\sigma_\infty(Q)$ such that*

$$N_{w^\dagger}(Q;B) = \sigma_\infty(Q)\mathfrak{S}(Q)B^2 + O_\varepsilon\Big(\frac{\|Q\|^{11+\varepsilon}}{A_1^6|\Delta_Q|^{1/2}}B^{3/2+\varepsilon}\Big),$$

where $\mathfrak{S}(Q)$ is given by (2.2)*, and $\sigma_\infty(Q)$ satisfies* (2.6).

Our final ingredient in the proof of Theorem 1 is a uniform upper bound for the singular series $\mathfrak{S}(Q)$. This will show, in particular, that for the family of quadratic forms (1.1) considered here, $\mathfrak{S}(Q)$ is convergent and actually grows rather slowly in terms of the coefficients of Q. The following result will be established in §7.

Proposition 3. *Let $n \geqslant 4$ and assume that Δ_Q is not a square when $n=4$. Then we have $\mathfrak{S}(Q) \ll_{\varepsilon,n} |\Delta_Q|^\varepsilon$.*

We are now ready to deduce the statement of Theorem 1 from the statements of Propositions 1–3. Let $\varepsilon > 0$, let $n \geqslant 4$, and assume that Δ_Q is not a square when $n=4$. On writing Q^σ for the diagonal quadratic form obtained by permuting the coefficients $A_1,\dots,A_n$, for each $\sigma \in S_n$, we deduce that

$$\begin{aligned} N(Q;B) &\ll_n 1 + \sum_{\sigma\in S_n}\sum_{j=0}^\infty N_{w^\dagger}(Q^\sigma;B/2^j) \\ &\ll_{\varepsilon,n} \Big(\frac{B^{n-2}}{m(Q)^{1/2}\|Q\|^{1/2}} + \frac{\|Q\|^{2n+3+\varepsilon}}{m(Q)^{3n/4+3}|\Delta_Q|^{1/2}}B^{(n-1+\delta_n)/2+\varepsilon}\Big)|\Delta_Q|^\varepsilon. \end{aligned}$$

This completes the proof of Theorem 1. The proof of Theorem 2 is handled in exactly the same way. Instead of using Proposition 1, however, we employ the main

technical result in recent joint work of the author with Dietmann [2, Proposition 1]. Once combined with Proposition 3, this latter result implies that

$$N_{w_Q}(Q;B) \ll_{\varepsilon,n} \Big(\frac{B^{n-2}}{|\Delta_Q|^{1/2}} + \|Q\|^{n/2+\varepsilon}B^{(n-1+\delta_n)/2+\varepsilon}\Big)|\Delta_Q|^{\varepsilon},$$

where

$$w_Q(\mathbf{x}) := w_0(2|A_1|^{1/2}x_1 - 2)w_0(|A_2|^{1/2}x_2)\cdots w_0(|A_n|^{1/2}x_n),$$

and w_0 is given by (2.3). Taking $B = X^{1/2}$, and arguing as in the deduction of Theorem 1, we therefore complete the proof of Theorem 2.

It is now time to recall the technical apparatus behind Heath-Brown's version of the Hardy–Littlewood circle method [6]. Recall the definitions (2.3) and (2.4) of the weight function $w_0 : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, and the constant c_0. Let $\omega(x) = 4c_0^{-1}w_0(4x-3)$, and define the function $h : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ by

$$h(x,y) = \sum_{j=1}^{\infty} \frac{1}{xj}\Big(\omega(xj) - \omega(|y|/xj)\Big).$$

It is shown in [6, §3] that $h(x,y)$ is infinitely differentiable for $(x,y) \in (0,\infty)\times\mathbb{R}$, and that $h(x,y)$ is non-zero only for $x \leqslant \max\{1, 2|y|\}$. Let Q be the quadratic form (1.1), let $w^\dagger$ be given by (2.5), and let $X > 1$. The kernel of our work is Heath-Brown's [6, Theorem 2]. For any $q \in \mathbb{N}$ and any $\mathbf{c} \in \mathbb{Z}^n$, we define the sum

$$S_q(\mathbf{c}) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{\mathbf{b} \,(\mathrm{mod}\ q)} e_q(aQ(\mathbf{b}) + \mathbf{b}.\mathbf{c}), \tag{2.7}$$

and the integral

$$I_q(\mathbf{c}) = \int_{\mathbb{R}^n} w^\dagger\Big(\frac{\mathbf{x}}{B}\Big)h\Big(\frac{q}{X}, \frac{Q(\mathbf{x})}{X^2}\Big)e_q(-\mathbf{c}.\mathbf{x})\mathrm{d}\mathbf{x}. \tag{2.8}$$

Then we deduce from the statements of [6, Theorems 1 and 2] that there exists a positive constant c_X, satisfying

$$c_X = 1 + O_N(X^{-N})$$

for any integer $N \geqslant 1$, such that

$$N_{w^\dagger}(Q;B) = c_X X^{-2} \sum_{\mathbf{c}\in\mathbb{Z}^n}\sum_{q=1}^{\infty} q^{-n}S_q(\mathbf{c})I_q(\mathbf{c}). \tag{2.9}$$

In our work we will make the choice

$$X = A_1^{1/2}B, \tag{2.10}$$

where as usual A_1 is assumed to be positive. Things can be made notationally less cumbersome by taking $X = B$ here instead. However, this would ultimately lead to a version of Propositions 1 and 2 with A_1 set to 1, and there is no extra technical difficulty in working with (2.10). In fact the key property required of X is that we should have $B^2X^{-2}\partial Q(\mathbf{x})/\partial x_1 \gg_n 1$ on the support of $w^\dagger$. When X is given by (2.10), we obviously have

$$B^2X^{-2}\partial Q(\mathbf{x})/\partial x_1 \geqslant 2A_1^{-1}A_1x_1 \gg 1$$

on $\mathrm{supp}(w^\dagger)$, which is satisfactory.

Our proof of Propositions 1 and 2 now has two major components: the estimation of the exponential sum (2.7) and that of the exponential integral (2.8). We will treat the former in §3, while the treatment of the latter is rather harder, and will be the focus of §4. We will deduce the statement of Proposition 1 in §5, and that of Proposition 2 in §6. Finally the proof of Proposition 3 will take place in §7.

3. Estimating $S_q(\mathbf{c})$

The purpose of this section is to provide good estimates for the exponential sums $S_q(\mathbf{c})$, as given by (2.7), which are uniform in the coefficients of Q. Much of this section follows the general lines of Heath-Brown's investigation [6, §§9–11]. A number of the results we will need may be quoted directly from that work, and we begin by recording the following multiplicativity property [6, Lemma 23].

Lemma 1. *If* $\gcd(u,v)=1$ *then*

$$S_{uv}(\mathbf{c})=S_u(\mathbf{c})S_v(\mathbf{c}).$$

In fact our work may be further simplified by appealing to the author's joint work with Dietmann [2], in which uniform estimates for the average order of $S_q(\mathbf{c})$ are provided for $n\geqslant 5$. The outcome of this investigation is the following result [2, Lemma 7].

Lemma 2. *Let* $n\geqslant 5$ *and let* $Y\geqslant 1$. *Then we have*

$$\sum_{q\leqslant Y}|S_q(\mathbf{c})|\ll_{\varepsilon,n}|\Delta_Q|^{1/2+\varepsilon}Y^{(n+3+\delta_n)/2+\varepsilon},$$

where δ_n *is given by* (1.7).

In view of Lemma 1, the function $q^{-n}S_q(\mathbf{0})$ is multiplicative. Moreover, Lemma 2 implies that the corresponding infinite sum $\sum_{q=1}^{\infty}q^{-n}S_q(\mathbf{0})$ is absolutely convergent for $n\geqslant 5$. Thus the usual analysis of the singular series yields

$$\sum_{q=1}^{\infty}q^{-n}S_q(\mathbf{0})=\prod_p\sum_{t=0}^{\infty}p^{-nt}S_{p^t}(\mathbf{0})=\prod_p\sigma_p, \tag{3.1}$$

where σ_p is given by (2.1), and we may conclude that

$$\sum_{q\leqslant Y}q^{-n}S_q(\mathbf{0})=\mathfrak{S}(Q)+O_{\varepsilon,n}\big(|\Delta_Q|^{1/2+\varepsilon}Y^{(3+\delta_n-n)/2+\varepsilon}\big), \tag{3.2}$$

for $n\geqslant 5$. Here, $\mathfrak{S}(Q)(Q)$ is given by (2.2).

It will suffice to assume that $n=4$ throughout the remainder of this section. The following easy upper bound for $S_q(\mathbf{c})$ follows from the proof of [6, Lemma 25].

Lemma 3. *We have*

$$S_q(\mathbf{c})\ll q^3\prod_{1\leqslant i\leqslant 4}\gcd(q,A_i)^{1/2}.$$

Proof. An application of Cauchy's inequality yields

$$|S_q(\mathbf{c})|^2\leqslant\phi(q)\sum_{\substack{a=1\\ \gcd(a,q)=1}}^{q}\ \sum_{\mathbf{d},\mathbf{e}=1}^{q}e_q\big(a(Q(\mathbf{d})-Q(\mathbf{e}))+\mathbf{c}.(\mathbf{d}-\mathbf{e})\big).$$

On substituting $\mathbf{d}=\mathbf{e}+\mathbf{f}$, we find that

$$e_q\big(a(Q(\mathbf{d})-Q(\mathbf{e}))+\mathbf{c}.(\mathbf{d}-\mathbf{e})\big)=e_q(aQ(\mathbf{f})+\mathbf{c}.\mathbf{f})e_q(a\mathbf{e}.\nabla Q(\mathbf{f})).$$

Since $\nabla Q(\mathbf{f}) = 2(A_1f_1,\dots,A_4f_4)$, the summation over $\mathbf{e}$ will produce a contribution of zero unless $q \mid A_if_i$ for $1 \leqslant i \leqslant 4$. This condition clearly holds for $\ll \gcd(q,A_1)\cdots\gcd(q,A_4)$ values of $\mathbf{f} \pmod q$, whence the result. $\square$

We will be able to improve upon Lemma 3 when q is square-free. The first step is to examine the sum at prime values of q. Define the quadratic form

$$Q^{-1}(\mathbf{y}) = A_1^{-1}y_1^2 + A_2^{-1}y_2^2 + A_3^{-1}y_3^2 + A_4^{-1}y_4^2,$$

with coefficients in $\mathbb{Q}$. When p is a prime such that $p \nmid 2\Delta_Q$ we may think of Q^{-1} as being defined modulo p. With this in mind, we have the following result.

Lemma 4. *Let p be an odd prime. Then we have*

$$S_p(\mathbf{c}) = \begin{cases} -\left(\frac{\Delta_Q}{p}\right)p^2, & \text{if } p \nmid Q^{-1}(\mathbf{c}),\\ \left(\frac{\Delta_Q}{p}\right)p^2(p-1), & \text{if } p \mid Q^{-1}(\mathbf{c}), \end{cases}$$

if $p \nmid \Delta_Q$, and

$$S_p(\mathbf{c}) \ll p^{5/2}\gcd(p,\Delta_Q Q^{-1}(\mathbf{c}))^{1/2}\prod_{i=1}^4 \gcd(p,A_i)^{1/2},$$

if $p \mid \Delta_Q$.

Proof. The first part follows on taking $n=4$ in [6, Lemma 26]. The second part follows by arguing along the lines of [2, Lemma 5]. $\square$

We may now combine Lemma 1 and Lemma 4 to provide an estimate for $S_q(\mathbf{c})$ in the case that q is square-free.

Lemma 5. *Let $q \in \mathbb{N}$ be square-free. Then we have*

$$S_q(\mathbf{c}) \ll_\varepsilon q^{5/2+\varepsilon}\gcd(q,\Delta_Q Q^{-1}(\mathbf{c}))^{1/2}\prod_{1\leqslant i\leqslant 4}\gcd(q,A_i)^{1/2}.$$

Proof. Since q is square-free we may write $q = 2^e\prod_{j=1}^r p_j$, with $p_1,\dots,p_r$ distinct odd primes and $e \in \{0,1\}$. Then it follows from Lemma 1, together with the trivial bound $|S_2(\mathbf{c})| \leqslant 2^4$, that

$$|S_q(\mathbf{c})| \leqslant 2^4\prod_{j=1}^r |S_{p_j}(\mathbf{c})|.$$

Now for each $1 \leqslant j \leqslant r$, it follows from Lemma 4 that

$$S_{p_j}(\mathbf{c}) \ll \begin{cases} p_j^{5/2}\gcd(p_j,\Delta_Q Q^{-1}(\mathbf{c}))^{1/2}, & \text{if } p_j \nmid \Delta_Q,\\ p_j^{5/2}\gcd(p_j,\Delta_Q Q^{-1}(\mathbf{c}))^{1/2}\prod_{i=1}^4\gcd(p_j,A_i)^{1/2}, & \text{if } p_j \mid \Delta_Q. \end{cases}$$

Putting these estimates together therefore yields the proof of Lemma 5. $\square$

We are now ready to discuss the average order of $|S_q(\mathbf{c})|$, as a function of q.

Lemma 6. *Let $Y \geqslant 1$. Then we have*

$$\sum_{q\leqslant Y}|S_q(\mathbf{c})| \ll_\varepsilon \begin{cases} |\Delta_Q|^{1/2+\varepsilon}|\mathbf{c}|^\varepsilon Y^{7/2+\varepsilon}, & \text{if } Q^{-1}(\mathbf{c}) \neq 0,\\ |\Delta_Q|^{1/2}Y^4, & \text{otherwise.} \end{cases}$$

Proof. The second bound is an easy consequence of Lemma 3. We therefore proceed under the assumption that $Q^{-1}(\mathbf{c}) \neq 0$. Write $q = uv$ for coprime u and v, such that u is square-free and v is square-full. Then we may combine Lemmas 1, 3 and 5 to deduce that

$$\begin{aligned} S_q(\mathbf{c}) &\ll |S_u(\bar{v}\mathbf{c})| v^3 \prod_{1\leqslant i\leqslant 4} \gcd(v, A_i)^{1/2} \\ &\ll_\varepsilon u^{5/2+\varepsilon} v^3 \gcd(u, \Delta_Q Q^{-1}(\mathbf{c}))^{1/2} \prod_{1\leqslant i\leqslant 4} \gcd(uv, A_i)^{1/2} \\ &\ll_\varepsilon |\Delta_Q|^{1/2} q^{5/2+\varepsilon} v^{1/2} \gcd(u, \Delta_Q Q^{-1}(\mathbf{c}))^{1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{q\leqslant Y} |S_q(\mathbf{c})| &\ll_\varepsilon |\Delta_Q|^{1/2} Y^{5/2+\varepsilon} \sum_{v\leqslant Y} v^{1/2} \sum_{u\leqslant Y/v} \gcd(u, \Delta_Q Q^{-1}(\mathbf{c})) \\ &\ll_\varepsilon |\Delta_Q|^{1/2+\varepsilon} |\mathbf{c}|^\varepsilon Y^{7/2+\varepsilon} \sum_{v\leqslant Y} v^{-1/2}, \end{aligned}$$

provided that $Q^{-1}(\mathbf{c}) \neq 0$. We complete the proof of Lemma 6 by noting that there are $O(V^{1/2})$ square-full values of $v \leqslant V$. □

Lemma 6 will suffice for our purposes if $Q^{-1}(\mathbf{c}) \neq 0$. To handle the case in which $Q^{-1}(\mathbf{c}) = 0$ we must work somewhat harder. Consider the Dirichlet series

$$D(s;\mathbf{c}) = \sum_{q=1}^{\infty} q^{-s} S_q(\mathbf{c}), \tag{3.3}$$

for $s = \sigma + it \in \mathbb{C}$. Then it follows from Lemma 6 that $D(s;\mathbf{c})$ is absolutely convergent for $\sigma > 4$. Moreover Lemma 1 yields $D(s;\mathbf{c}) = \prod_p D_p(s;\mathbf{c})$, where

$$D_p(s;\mathbf{c}) = \sum_{k=0}^{\infty} p^{-ks} S_{p^k}(\mathbf{c}). \tag{3.4}$$

We now investigate the factors $D_p(s;\mathbf{c})$ more carefully, for which we must distinguish between whether or not p is a divisor of $2\Delta_Q$.

Suppose first that $p \mid 2\Delta_Q$. Then one easily deduces from Lemma 3 that

$$\begin{aligned} D_p(s;\mathbf{c}) &\ll \sum_{k=0}^{\infty} p^{(3-\sigma)k} p^{(\min\{\nu_p(A_1),k\}+\cdots+\min\{\nu_p(A_4),k\})/2} \\ &\ll \sum_{k=0}^{\infty} p^{(7/2-\sigma)k} p^{\nu_p(\Delta_Q)/2} \ll p^{\nu_p(\Delta_Q)/2}, \end{aligned}$$

if $\sigma > 7/2$, where $\nu_p(n)$ denotes the p-adic order of any non-zero integer n. Hence

$$\prod_{p\mid 2\Delta_Q} D_p(s;\mathbf{c}) \ll_\varepsilon |\Delta_Q|^{1/2+\varepsilon}. \tag{3.5}$$

Suppose now that $p \nmid 2\Delta_Q$. Then it follows from Lemmas 3 and 4 that

$$D_p(s;\mathbf{c}) = 1 + \Big(\frac{\Delta_Q}{p}\Big) p^{2-s}(p-1) + O_\delta(p^{-1-2\delta}),$$

for $\sigma \geqslant 7/2+\delta$, since $Q^{-1}(\mathbf{c})=0$. On writing $\chi_Q(p)=(\frac{\Delta_Q}{p})$, we therefore deduce that

$$D_p(s;\mathbf{c}) = \big(1-\chi_Q(p)p^{3-s}\big)^{-1}\big(1+O_\delta(p^{-1-\delta})\big), \tag{3.6}$$

for $\sigma \geqslant 7/2+\delta$. We may combine this with (3.5) to conclude that

$$D(s;\mathbf{c}) = L(s-3,\chi_Q)E(s;\mathbf{c}), \tag{3.7}$$

in this region, where $E(s;\mathbf{c}) \ll_{\delta,\varepsilon} |\Delta_Q|^{1/2+\varepsilon}$. In particular $D(s;\mathbf{c})$ has an analytic continuation to the half-plane $\sigma > 7/2$.

Let Y be half an odd integer. Then it follows from an application of Perron's formula (see the proof of Titchmarsh [10, Lemma 3.12], for example), together with the second estimate in Lemma 6, that

$$\sum_{q\leqslant Y} S_q(\mathbf{c}) = \frac{1}{2\pi i}\int_{6-iT}^{6+iT} \frac{D(s;\mathbf{c})Y^s}{s}\mathrm{d}s + O\Big(\frac{|\Delta_Q|^{1/2}Y^6}{T}\Big), \tag{3.8}$$

for any $T\geqslant 1$. Let $\alpha = 7/2+\varepsilon$. Then we proceed to move the line of integration back to $\sigma=\alpha$. Now (3.7) yields

$$D(\sigma+it;\mathbf{c}) \ll_\varepsilon |\Delta_Q|^{1/2+\varepsilon}|L(\sigma-3+it,\chi_Q)|, \tag{3.9}$$

for $\sigma\in[\alpha,6]$. We now require the following simple upper bound for the size of the Dirichlet L-function.

Lemma 7. *Let χ be a Dirichlet character modulo k. Then we have*

$$L(\sigma+it,\chi) \ll_{\varepsilon,\sigma} \begin{cases} k^{(1-\sigma)/2+\varepsilon}\tau^{1-\sigma+\varepsilon}, & \text{if } \sigma\in(1/2,1] \text{ and } \chi \text{ non-principal},\\ 1, & \text{if } \sigma\in(1,\infty),\end{cases}$$

where $\tau = 1+|t|$.

Proof. The result is trivial for $\sigma>1$. Assuming that $\sigma\in(1/2,1]$, therefore, we may combine the Pólya–Vinogradov inequality with partial summation, to obtain

$$\begin{aligned} L(\sigma+it,\chi) &\ll_\sigma \sum_{n\leqslant x}\frac{1}{n^\sigma} + (1+|t|)\int_x^\infty \frac{k^{1/2}\log k}{u^{\sigma+1}}\mathrm{d}u \\ &\ll_\sigma x^{1-\sigma}\log x + \frac{(1+|t|)k^{1/2}\log k}{x^\sigma}, \end{aligned}$$

for any $x\geqslant 1$. The proof of the lemma is completed by taking $x=(1+|t|)k^{1/2}$ and noting that $\log z \ll_\varepsilon z^\varepsilon$ for any $z\geqslant 1$. $\square$

Sharper versions of Lemma 7 are available in the literature, although we will not need anything so deep here. For example, Heath-Brown [5] has shown that $L(\sigma+it,\chi)\ll_{\varepsilon,\sigma}(k\tau)^{3(1-\sigma)/8+\varepsilon}$, for $\sigma\in(1/2,1]$, where $\tau=1+|t|$ and χ is any non-principal character modulo k.

We continue with our analysis of the Dirichlet series $D(s;\mathbf{c})$. Now χ_Q is a non-principal character, since Δ_Q is not a square when $n=4$. Hence applying Lemma 7 in (3.9) yields

$$D(\sigma+it;\mathbf{c}) \ll_\varepsilon |\Delta_Q|^{1/2+\varepsilon}\begin{cases} |\Delta_Q|^{(4-\sigma)/2+\varepsilon}(1+|t|)^{4-\sigma+\varepsilon}, & \sigma\in[\alpha,4],\\ 1, & \sigma\in(4,6].\end{cases}$$

In particular

$$\int_{\alpha\pm iT}^{6\pm iT}\frac{D(s;\mathbf{c})Y^s}{s}\mathrm{d}s \ll_\varepsilon |\Delta_Q|^{1/2+\varepsilon}\Big(|\Delta_Q|^{1/4}\frac{Y^{7/2+\varepsilon}T^\varepsilon}{T^{1/2}} + \frac{Y^6}{T}\Big).$$

Turning to the contribution from the vertical lines, we will employ the mean-value estimate
$$\int_0^U |L(\sigma+it,\chi)|^2 \mathrm{d}t \ll_\sigma k^{1/2}U,$$
that is valid for any $\sigma \in (1/2,1)$ and any character modulo k. But then it follows from this, together with an application of (3.9) and Cauchy's inequality, that
$$\begin{aligned}\int_{\alpha-iT}^{\alpha+iT} \frac{D(s;\mathbf{c})Y^s}{s}\mathrm{d}s &\ll_\varepsilon |\Delta_Q|^{1/2+\varepsilon}Y^{7/2+\varepsilon}\int_0^T \frac{|L(1/2+\varepsilon+it,\chi_Q)|}{1+t}\mathrm{d}t\\ &\ll_\varepsilon |\Delta_Q|^{3/4+\varepsilon}Y^{7/2+\varepsilon}T^\varepsilon.\end{aligned}$$
We are now in a position to bring this all together in (3.8). Thus we conclude the proof of the following result by taking $T=Y^{5/2}$, and noting that $D(s;\mathbf{c})Y^s/s$ is holomorphic in the half-plane $\sigma \geqslant \alpha$.

Lemma 8. *Suppose that Δ_Q is not a square and $Q^{-1}(\mathbf{c})=0$. Then we have*
$$\sum_{q\leqslant Y} S_q(\mathbf{c}) \ll_\varepsilon |\Delta_Q|^{3/4+\varepsilon}Y^{7/2+\varepsilon}.$$

We conclude this section with a few words about the sum $\sum_{q\leqslant Y} q^{-n}S_q(\mathbf{0})$ in the case $n=4$. In the notation of (3.3), we have $\sum_{q=1}^\infty q^{-4}S_q(\mathbf{0}) = D(4;\mathbf{0})$, and the argument used to prove Lemma 8 ensures that $D(4;\mathbf{0})$ is convergent. Thus (3.1) continues to hold when $n=4$. Moreover, we may trace through our application of Perron's formula to conclude that
$$\sum_{q\leqslant Y} q^{-4}S_q(\mathbf{0}) = \mathfrak{S}(Q) + O_\varepsilon\big(|\Delta_Q|^{3/4+\varepsilon}Y^{-1/2+\varepsilon}\big). \tag{3.10}$$

4. Estimating $I_q(\mathbf{c})$

Let $q\in\mathbb{N}$, let $\mathbf{c}\in\mathbb{Z}^n$ and recall the definition (1.1) of the quadratic form Q. We continue to employ the notation $\|Q\|$ for the height of Q, as given by (1.3), and the convention that $A_1>0$ in (1.1). The goal of this section is to study the integral (2.8). In fact it will be convenient to investigate the behaviour of the integral
$$I_q(\mathbf{c};w) = \int_{\mathbb{R}^n} w\Big(\frac{\mathbf{x}}{B}\Big)h\Big(\frac{q}{X},\frac{Q(\mathbf{x})}{X^2}\Big)e_q(-\mathbf{c}.\mathbf{x})\mathrm{d}\mathbf{x}, \tag{4.1}$$
for a rather general class of weight functions $w:\mathbb{R}^n\to\mathbb{R}_{\geqslant 0}$. Our first task is therefore to define the class $\mathcal{W}(S)$ of weight functions that we will work with. Here, S is an arbitrary set of parameters that we always assume to contain n.

Our presentation will be much along the lines of [6, §§2,6]. By a weight function w, we will henceforth mean a non-negative function $w:\mathbb{R}^n\to\mathbb{R}_{\geqslant 0}$, which is infinitely differentiable and has compact support. Given such a function w, we let $\dim(w)=n$ denote the dimension of the domain of w and $\mathrm{Rad}(w)$ be the smallest R such that w is supported in the hypercube $[-R,R]^n$. Moreover for each integer $j\geqslant 0$ we let
$$\kappa_j(w) = \max\Big\{\Big|\frac{\partial^{j_1+\cdots+j_n}w(\mathbf{x})}{\partial^{j_1}x_1\cdots\partial^{j_n}x_n}\Big| : \mathbf{x}\in\mathbb{R}^n,\ j_1+\cdots+j_n=j\Big\}.$$
We define $\mathcal{W}_1(S)$ to be the set of weight functions $w:\mathbb{R}^n\to\mathbb{R}_{\geqslant 0}$, such that
$$\dim(w), \mathrm{Rad}(w), \kappa_0(w), \kappa_1(w), \dots$$

are all bounded by corresponding quantities involving parameters from the set S. In particular it is clear that $w^\dagger \in \mathcal{W}_1(n)$, where $w^\dagger$ is given by (2.5). We now specify the set of functions $\mathcal{W}(S) \subset \mathcal{W}_1(S)$. Given $w \in \mathcal{W}_1(S)$, we will say that $w \in \mathcal{W}(S)$ if $x_1 \gg_S 1$ on $\mathrm{supp}(w)$. In particular it then follows that there is exactly one solution to the equation $Q(x_1, \mathbf{y}) = 0$, for given $\mathbf{y} = (x_2, \dots, x_n) \in \mathbb{R}^n$ such that

$$\Big(\sqrt{-A_1^{-1}(A_2x_2^2 + \cdots + A_nx_n^2)}, \mathbf{y}\Big) \in \mathrm{supp}(w).$$

We conclude our discussion of the class $\mathcal{W}(S)$ by noting that $w^\dagger \in \mathcal{W}(n)$.

Returning to the task of estimating $I_q(\mathbf{c}) = I_q(\mathbf{c}; w^\dagger)$, it will clearly suffice to estimate (4.1) for any choice of weight $w \in \mathcal{W}(S)$. On recalling that $X = A_1^{1/2}B$ in (2.10), a simple change of variables yields

$$I_q(\mathbf{c}; w) = B^n \int_{\mathbb{R}^n} w(\mathbf{x}) h(A_1^{-1/2}B^{-1}q, R(\mathbf{x})) e_q(-B\mathbf{c}.\mathbf{x}) \mathrm{d}\mathbf{x},$$

where

$$R = A_1^{-1}Q \in \mathbb{Q}[\mathbf{x}].$$

In particular $R(\mathbf{x}) \ll_S \|Q\|/A_1$ for any $\mathbf{x} \in \mathrm{supp}(w)$. It follows from the properties of h discussed in §2, together with the definition of the set $\mathcal{W}(S)$, that $I_q(\mathbf{c})$ will vanish unless $q \ll_S B\|Q\|/A_1^{1/2}$. Following Heath-Brown we proceed by defining

$$I_r^*(\mathbf{v}; w) = \int_{\mathbb{R}^n} w(\mathbf{x}) h(r, R(\mathbf{x})) e_r(-\mathbf{v}.\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{4.2}$$

so that

$$I_q(\mathbf{c}; w) = B^n I_r^*(\mathbf{v}; w), \tag{4.3}$$

with $r = A_1^{-1/2}B^{-1}q$ and $\mathbf{v} = A_1^{-1/2}\mathbf{c}$. We note that

$$\frac{\partial I_q(\mathbf{c}; w)}{\partial q} = \frac{B^{n-1}}{A_1^{1/2}} \frac{\partial I_r^*(\mathbf{v}; w)}{\partial r}. \tag{4.4}$$

In our work we will need good upper bounds for the integral $I_r^*(\mathbf{v}; w)$ and its first derivative with respect to r, that are uniform in the coefficients of Q. For this purpose it will clearly suffice to assume that $q \ll_S B\|Q\|/A_1^{1/2}$, or equivalently that $r \ll_S \|Q\|/A_1$. Following Heath-Brown's argument in [6, §7], let $\mathcal{H}$ denote the set of infinitely differentiable functions $f : (0, \infty) \times \mathbb{R} \to \mathbb{C}$ such that for each $N \in \mathbb{N}$ there exist absolute constants $K_{j,N} > 0$ for which

$$\Big|\frac{\partial^j f(r, y)}{\partial y^j}\Big| \leqslant \begin{cases} K_{0,N}\big(r^N + \min\{1, (r/|y|)^N\}\big), & \text{if } j = 0, \\ K_{j,N} r^{-j} \min\{1, (r/|y|)^N\}, & \text{if } j \geqslant 1. \end{cases} \tag{4.5}$$

Define the integral

$$J_r(\mathbf{u}; \omega, f) = \int_{\mathbb{R}^n} \omega(\mathbf{x}) f(r, R(\mathbf{x})) e(-\mathbf{u}.\mathbf{x}) \mathrm{d}\mathbf{x},$$

for any $\omega \in \mathcal{W}(S)$ and any $f \in \mathcal{H}$. Here r is restricted to the interval $(0, \infty)$ and $\mathbf{u}$ can be any vector in $\mathbb{R}^n$. Let $k \in \{0, 1\}$. Then a straightforward examination of the proof of [6, Lemma 14] reveals that there exists $\omega^{(k)} \in \mathcal{W}(S)$ and $f^{(k)} \in \mathcal{H}$ such that $\mathrm{supp}(\omega^{(k)}) \subseteq \mathrm{supp}(w)$ and

$$\frac{\partial^k I_r^*(\mathbf{v}; w)}{\partial r^k} \ll r^{-1-k} |J_r(r^{-1}\mathbf{v}; \omega^{(k)}, f^{(k)})|. \tag{4.6}$$

Indeed the only thing to check here is that the statement of [6, Lemma 14] remains valid when one starts with an arbitrary weight function belonging to $\mathcal{W}(S)$, and that it produces auxiliary weight functions also belonging to $\mathcal{W}(S)$.

In view of (4.6) it will now be enough to estimate $J_r(\mathbf{u};\omega,f)$ for given $\omega\in\mathcal{W}(S)$ and $f\in\mathcal{H}$. We begin by recording a rather trivial upper bound for this integral. The following result is established much as in [6, Lemma 15].

Lemma 9. *Let $\omega\in\mathcal{W}(S)$ and let $f\in\mathcal{H}$. Then we have $J_r(\mathbf{u};\omega,f)\ll_S r$.*

Proof. Since $f\in\mathcal{H}$, we may deduce from (4.5) that

$$J_r(\mathbf{u};\omega,f)\ll_N \int_{\mathbb{R}^n}\omega(\mathbf{x})\Big(r^N+\min\Big\{1,\frac{r^N}{|R(\mathbf{x})|^N}\Big\}\Big)\mathrm{d}\mathbf{x},$$

for any $N\geqslant 1$. If $r\geqslant 1$ then we may take $N=1$ in this estimate to deduce that $J_r(\mathbf{u};\omega,f)\ll_S 1+r\ll_S r$, which is satisfactory for the lemma. If $r<1$ then we take $N=2$ to obtain

$$J_r(\mathbf{u};\omega,f)\ll \int_{\mathbb{R}^n}\omega(\mathbf{x})\Big(r+\min\Big\{1,\frac{r^2}{R(\mathbf{x})^2}\Big\}\Big)\mathrm{d}\mathbf{x}.$$

Since $x_1\gg_S 1$ for any $\mathbf{x}\in\mathrm{supp}(\omega)$ we have

$$\frac{\partial R}{\partial x_1}\gg_S 1, \tag{4.7}$$

on $\mathrm{supp}(\omega)$. On substituting $y=R(\mathbf{x})$ for x_1 we therefore obtain

$$J_r(\mathbf{u};\omega,f)\ll_S r+\int_{|y|\leqslant r} I(y;\omega)\mathrm{d}y+r^2\int_{|y|>r}\frac{I(y,\omega)}{y^2}\mathrm{d}y,$$

where

$$I(y)=I(y;\omega)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\omega(\mathbf{x})\frac{\mathrm{d}x_2\cdots\mathrm{d}x_n}{\partial R/\partial x_1}, \tag{4.8}$$

in which x_1 is defined by the relation $y=R(\mathbf{x})$. We proceed to show that $I\in\mathcal{W}_1(S)$, with $\dim(I)=1$. To see this it suffices to check that

$$\frac{\partial^j}{\partial y^j}\frac{\omega(x_1(y),x_2,\ldots,x_n)}{\partial R(x_1(y),x_2,\ldots,x_n)/\partial x_1}\ll_S 1$$

on $\mathrm{supp}(\omega)$, for any $j\geqslant 0$. But this follows from the lower bound (4.7) and the fact that $\partial x_1/\partial y=(\partial R/\partial x_1)^{-1}$. Having established that $I\in\mathcal{W}_1(S)$, we obtain $J_r(\mathbf{u};\omega,f)\ll_S r$ when $r<1$, as required to complete the proof of Lemma 9. $\square$

Turning to a more sophisticated treatment of $J_r(\mathbf{u};\omega,f)$, for given $\omega\in\mathcal{W}(S)$ and $f\in\mathcal{H}$, we define

$$K=A_1^{-1}\|Q\|,\quad K(\omega)=nK\mathrm{Rad}(\omega)^2. \tag{4.9}$$

Then it is clear that $K\ll_S K(\omega)\ll_S K$ for any $\omega\in\mathcal{W}(S)$, and $|R(\mathbf{x})|\leqslant K(\omega)$ for any $\mathbf{x}\in\mathrm{supp}(\omega)$. Recall the definition (2.3) of w_0, and define

$$\omega_2(v)=w_0\Big(\frac{v}{2K(\omega)}\Big),\quad \omega_1(\mathbf{x})=\frac{\omega(\mathbf{x})}{\omega_2(R(\mathbf{x}))}.$$

Then it is not hard to check that ω_2 has compact support $[-2K(\omega),2K(\omega)]$, and that $\omega_1\in\mathcal{W}(S)$, with $\mathrm{supp}(\omega_1)\subseteq\mathrm{supp}(\omega)$. An examination of the proof of [6,

Lemma 17] reveals that

$$J_r(\mathbf{u};\omega,f)=\int_{-\infty}^{\infty}p(t)\int_{\mathbb{R}^n}\omega_1(\mathbf{x})e(tR(\mathbf{x})-\mathbf{u}.\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{4.10}$$

with

$$p(t)=\int_{-\infty}^{\infty}\omega_2(v)f(r,v)e(-tv)\mathrm{d}v.$$

We proceed to establish the bound

$$p(t)\ll_{N,S} Kr(r|t|)^{-N}, \tag{4.11}$$

for any $N\geqslant 0$, where K is given by (4.9). Writing $g(v)=\omega_2(v)f(r,v)$, a repeated application of integration by parts reveals that

$$p(t)\ll_{N,S}|t|^{-N}\int_{-2K(\omega)}^{2K(\omega)}\Big|\frac{d^N g(v)}{dv^N}\Big|\mathrm{d}v,$$

for any $N\geqslant 0$. But on employing the inequalities (4.5) satisfied by $f(r,v)$, together with the fact that $r\ll_S K$, we easily deduce that for any $M\geqslant 1$ and $N\geqslant 0$ we have

$$\begin{aligned}\frac{d^N g(v)}{dv^N}&\ll_{M,N,S}\frac{r^M+\min\{1,(r/|v|)^M\}}{K^N}+\frac{\min\{1,(r/|v|)^2\}}{r^N}\\&\ll_{N,S}\begin{cases} r^{1-N}+r^{-N}\min\{1,(r/|v|)^2\}, & \text{if } r<1,\\ r^{1-N}, & \text{if } r\geqslant 1.\end{cases}\end{aligned}$$

It is now straightforward to deduce the estimate in (4.11).

We are now ready to use the above analysis to deduce a series of useful basic estimates for the integral $I_q(\mathbf{c};w)$, for any $w\in\mathcal{W}(S)$. Our first result in this direction will be used to show that large values of $\mathbf{c}$ make a negligible contribution in our analysis.

Lemma 10. *Let $\mathbf{c}\in\mathbb{Z}^n$ with $\mathbf{c}\neq\mathbf{0}$. Then we have*

$$I_q(\mathbf{c};w)\ll_{N,S}\frac{B^{n+1}}{q}\frac{\|Q\|^{N+1}}{A_1^{N/2+1/2}|\mathbf{c}|^N},$$

for any $N\geqslant 0$.

Proof. The proof of Lemma 10 closely follows the proof of [6, Lemma 19]. Recall (4.2), (4.3) and the definition (4.9) of K. Then in order to establish Lemma 10 it will suffice to show that

$$I_r^*(\mathbf{v};w)\ll_{N,S}K^{N+1}r^{-1}|\mathbf{v}|^{-N}, \tag{4.12}$$

for any $N\geqslant 0$. To deduce (4.12) we employ the identity (4.10) and the estimate (4.11) for $p(t)$. Suppose first that $|\mathbf{u}|\gg_S K|t|$, where K is given by (4.9). Then we apply [6, Lemma 10] with $f(\mathbf{x})=tR(\mathbf{x})-\mathbf{u}.\mathbf{x}$ and $\lambda=|\mathbf{u}|$. This gives

$$\int_{\mathbb{R}^n}\omega_1(\mathbf{x})e(tR(\mathbf{x})-\mathbf{u}.\mathbf{x})\mathrm{d}\mathbf{x}\ll_{M,S}|\mathbf{u}|^{-M},$$

for any $M\geqslant 1$. Once inserted into (4.10), and combined with an application of (4.11) with $N=0$, we obtain a contribution of

$$\ll_{M,S}Kr\int_{|t|\ll_S K^{-1}|\mathbf{u}|}|\mathbf{u}|^{-M}\mathrm{d}t\ll_{M,S}r|\mathbf{u}|^{1-M}$$

to $J_r(\mathbf{u};\omega,f)$. When $|\mathbf{u}| \ll_S K|t|$ we use the trivial bound

$$\int_{\mathbb{R}^n} \omega_1(\mathbf{x}) e(tR(\mathbf{x}) - \mathbf{u}.\mathbf{x}) \mathrm{d}\mathbf{x} \ll_S 1,$$

and take $N = M$ in (4.11). This contributes

$$\ll_{M,S} Kr^{1-M} \int_{|t| \gg K^{-1}|\mathbf{u}|} |t|^{-M} \mathrm{d}t \ll_{M,S} K^M r^{1-M} |\mathbf{u}|^{1-M}$$

to $J_r(\mathbf{u};\omega,f)$. We may now combine these estimates in (4.10) to conclude that

$$J_r(\mathbf{u};\omega,f) \ll_{M,S} r|\mathbf{u}|^{1-M} + K^M r^{1-M} |\mathbf{u}|^{1-M} \ll_{M,S} K^M r^{1-M} |\mathbf{u}|^{1-M},$$

for any $M \geqslant 1$, since $r \ll_S K$. Thus it follows that

$$J_r(\mathbf{u};\omega,f) \ll_{N,S} K^{N+1} r^{-N} |\mathbf{u}|^{-N}$$

for any $N \geqslant 0$. We may insert this into (4.6) with $k = 0$ to deduce that (4.12) holds for any $N \geqslant 0$. This completes the proof of Lemma 10. $\square$

We will need a finer estimate for $I_q(\mathbf{c}; w)$ when $\mathbf{c}$ has small modulus. The following result is established along the lines of [6, Lemma 22].

Lemma 11. *Let $\mathbf{c} \in \mathbb{Z}^n$ with $\mathbf{c} \neq \mathbf{0}$. Then we have*

$$q^k \frac{\partial^k I_q(\mathbf{c}; w)}{\partial q^k} \ll_{\varepsilon,S} \frac{\|Q\|^{3n/2+1+\varepsilon}}{A_1^{n/2+1}|\Delta_Q|} B^{n/2+1+\varepsilon} \Big(\frac{|\mathbf{c}|}{q}\Big)^{1-n/2+\varepsilon},$$

for $k \in \{0, 1\}$.

Proof. Our starting point arises from (4.3), (4.4) and (4.6), which render it sufficient to study the quantity $J_r(\mathbf{u};\omega,f)$ for $\omega \in \mathcal{W}(S)$ and $f \in \mathcal{H}$. We will use the identity (4.10), where as stated there $\omega_1 \in \mathcal{W}(S)$ is such that $\mathrm{supp}(\omega_1) \subseteq \mathrm{supp}(\omega)$, and $p(t)$ satisfies (4.11) for any $N \geqslant 0$.

It will be convenient to introduce parameters $\delta > 0$ and $T \geqslant 1$, to be selected in due course. On recalling the definition (4.9) of the quantity K, our immediate goal is to estimate $J_r(\mathbf{u};\omega,f)$ under the assumption that

$$|\mathbf{u}| \geqslant KT^2. \tag{4.13}$$

We proceed by using the subdivision process detailed in [6, Lemma 2] to split up the range for $\mathbf{x}$. On combining this result with (4.10) it therefore follows that

$$J_r(\mathbf{u};\omega,f) = \delta^{-n} \int_{-\infty}^{\infty} p(t) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega_\delta\Big(\frac{\mathbf{x}-\mathbf{y}}{\delta}, \mathbf{y}\Big) e(tR(\mathbf{x}) - \mathbf{u}.\mathbf{x}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}t,$$

where $F(\mathbf{x}) = \omega_\delta(\delta^{-1}(\mathbf{x}-\mathbf{y}), \mathbf{y})$ belongs to $\mathcal{W}(S)$ and has $\mathrm{supp}(F) \subseteq \mathrm{supp}(\omega_1) \subseteq \mathrm{supp}(\omega)$. Writing $\mathbf{x} = \mathbf{y} + \delta\mathbf{z}$ we deduce that

$$|J_r(\mathbf{u};\omega,f)| \leqslant \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |p(t)| \Big| \int_{\mathbb{R}^n} \omega_3(\mathbf{z}) e(tR(\mathbf{x}) - \mathbf{u}.\mathbf{x}) \mathrm{d}\mathbf{z} \Big| \mathrm{d}t\mathrm{d}\mathbf{y}, \tag{4.14}$$

with $\omega_3(\mathbf{z}) = \omega_3(\mathbf{z},\mathbf{y}) = \omega_\delta(\mathbf{z},\mathbf{y}) \in \mathcal{W}(S)$ and $\mathrm{supp}(\omega_3) \subseteq \mathrm{supp}(\omega)$. In order to effectively estimate the inner integral in (4.14), we must now differentiate the pairs $(\mathbf{y}, t)$ according to whether or not they yield a negligible estimate. We will say that the pair $(\mathbf{y}, t)$ is 'good' if

$$\delta|t\nabla R(\mathbf{y}) - \mathbf{u}| \geqslant T \max\{1, K|t|\delta^2\}, \tag{4.15}$$

and 'bad' otherwise.

We begin by treating the case of good pairs $(\mathbf{y}, t)$. Working under the assumption (4.13) and (4.15), we will estimate the inner integral in (4.14) via an application of [6, Lemma 10]. Let

$$f(\mathbf{z}) = tR(\mathbf{y} + \delta\mathbf{z}) - \mathbf{u}.\mathbf{y} - \delta\mathbf{u}.\mathbf{z}.$$

Then the partial derivatives of order at least two are all $O_S(K|t|\delta^2)$, and furthermore

$$|\nabla f(\mathbf{z})| = \delta|t\nabla R(\mathbf{y}) - \mathbf{u}| + O_S(K|t|\delta^2).$$

But then $|\nabla f(\mathbf{z})| \gg_S T \max\{1, K|t|\delta^2\}$, and it is easy to deduce that

$$\int_{\mathbb{R}^n} \omega_3(\mathbf{z}) e(tR(\mathbf{x}) - \mathbf{u}.\mathbf{x}) \mathrm{d}\mathbf{z} \ll_{N,S} T^{-N},$$

for any $N \geqslant 1$. In view of (4.11) and the fact that all of the relevant $\mathbf{y}$ satisfy $|\mathbf{y}| \ll_S 1$, we therefore obtain an overall contribution of $O_{N,S}(KT^{-N})$ to $J_r(\mathbf{u}; \omega, f)$ from the good pairs in (4.14).

Turning to the contribution from the bad pairs, we henceforth set

$$\delta = K^{-1/2}|\mathbf{u}|^{-1/2}. \tag{4.16}$$

Then it follows from (4.15) that

$$|t\nabla R(\mathbf{y}) - \mathbf{u}| \leqslant K^{1/2}T|\mathbf{u}|^{1/2} \max\{1, |t|/|\mathbf{u}|\},$$

if $(\mathbf{y}, t)$ is a bad pair. We claim that if $(\mathbf{y}, t)$ is a bad pair then

$$K^{-1}|\mathbf{u}| \ll_S |t| \ll_S |\mathbf{u}|, \tag{4.17}$$

and

$$|t\nabla R(\mathbf{y}) - \mathbf{u}| \ll_S K^{1/2}T|\mathbf{u}|^{1/2}. \tag{4.18}$$

It will clearly suffice to establish (4.17), since (4.18) is a trivial consequence of this and the previous inequality. Suppose first that $|t| \leqslant |\mathbf{u}|$. Then $t\nabla R(\mathbf{y}) = \mathbf{u} + O_S(K^{1/2}T|\mathbf{u}|^{1/2})$, and so

$$K|t| \gg_S |t\nabla R(\mathbf{y})| \gg_S |\mathbf{u}|,$$

since (4.13) implies $|\mathbf{u}| \gg_S K^{1/2}T|\mathbf{u}|^{1/2}$. This establishes (4.17) in this case. Suppose now that $|t| \geqslant |\mathbf{u}|$, so that $\mathbf{u}/t = \nabla R(\mathbf{y}) + O_S(K^{1/2}T|\mathbf{u}|^{-1/2})$. Recall the definition of the weight function

$$\omega_3(\mathbf{z}) = \omega_\delta(\mathbf{z}, \mathbf{y}) = c_0^{-n}\omega_1(\mathbf{x}) \prod_{i=1}^n w_0\Big(\frac{x_i - y_i}{\delta}\Big),$$

as it is constructed in the proof of [6, Lemma 2]. In particular it follows that we must have $x_i - \delta \leqslant y_i \leqslant x_i + \delta$ for $1 \leqslant i \leqslant n$, if $\omega_3(\mathbf{z})$ is to be non-zero in (4.14). Hence $|\nabla R(\mathbf{y})| \gg_S 1 - \delta \gg_S 1 \geqslant K^{1/2}T|\mathbf{u}|^{-1/2}$, by (4.13). Thus we must have $|\mathbf{u}| \gg_S |t|$ under the assumption that $T \gg_S 1$, and so (4.17) holds in this case also.

Drawing all of this together we deduce from (4.14) that

$$J_r(\mathbf{u}; \omega, f) \ll_{N,S} KT^{-N} + \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |p(t)| \int_{\mathbb{R}^n} |\omega_3(\mathbf{z})| \mathrm{d}\mathbf{z}\mathrm{d}t\mathrm{d}\mathbf{y},$$

provided that (4.13) holds, where $(\mathbf{y}, t)$ runs over values of $\mathbb{R}^{n+1}$ such that (4.17) and (4.18) hold, with $|\mathbf{y}| \ll_S 1$. We now substitute $\mathbf{x} = \mathbf{y} + \delta\mathbf{z}$ for $\mathbf{y}$ in this estimate. Now it is clear from (4.16) that

$$|t\nabla R(\mathbf{x}) - t\nabla R(\mathbf{y})| \ll_S K|t|\delta \ll_S K^{1/2}|\mathbf{u}|^{1/2} \leqslant K^{1/2}T|\mathbf{u}|^{1/2},$$

since $|\mathbf{z}| \ll_S 1$ and $|t| \ll_S |\mathbf{u}|$. Thus if $\mathbf{y}$ satisfies (4.18) then so must $\mathbf{x}$. On employing the bound $p(t) \ll_S Kr$ that follows from (4.11), we therefore conclude that there exists $t \in \mathbb{R}$ in the range (4.17) such that

$$J_r(\mathbf{u};\omega,f) \ll_{N,S} KT^{-N} + Kr|\mathbf{u}|\mathrm{Vol}(\mathcal{S}_t),$$

for any $N \geqslant 1$, where

$$\mathcal{S}_t = \{\mathbf{x} \in \mathrm{supp}(\omega) : \; |t\nabla R(\mathbf{x}) - \mathbf{u}| \ll_S K^{1/2}T|\mathbf{u}|^{1/2}\}.$$

An easy calculation reveals that each x_i in $\mathcal{S}_t$ is restricted to an interval of length $O_S(|A_i|^{-1}A_1K^{3/2}T|\mathbf{u}|^{-1/2})$, whence

$$\mathrm{Vol}(\mathcal{S}_t) \ll_S \frac{K^{3n/2}A_1^n}{|\Delta_Q|}|\mathbf{u}|^{-n/2}T^n.$$

It therefore follows that

$$J_r(\mathbf{u};\omega,f) \ll_{N,S} KT^{-N} + \frac{K^{3n/2+1}A_1^n}{|\Delta_Q|}r|\mathbf{u}|^{1-n/2}T^n, \tag{4.19}$$

for any $N \geqslant 1$, under the assumption that (4.13) holds.

In order to complete the proof of Lemma 11 we let $\varepsilon \in (0, 1/2)$ and recall the definition (4.9) of K. We will show that

$$J_r(\mathbf{u};\omega,f) \ll_{\varepsilon,S} \frac{K^{3n/2+1}A_1^n}{|\Delta_Q|}r|\mathbf{u}|^{1-n/2}(r^{-1}|\mathbf{u}|)^{\varepsilon}. \tag{4.20}$$

Suppose first that $|\mathbf{u}| \leqslant K^{(n+2\varepsilon)/n}r^{-2\varepsilon/n}$. Then we have

$$K^{(n+2\varepsilon)/n}r^{-2\varepsilon/n} \ll_S K^{n/(n-2-2\varepsilon)}r^{-2\varepsilon/(n-2-2\varepsilon)},$$

since $r \ll_S K$ and $K \geqslant 1$. Hence it follows from Lemma 9 that

$$J_r(\mathbf{u};\omega,f) \ll_S r \leqslant Kr \ll_S K^{n/2+1}r|\mathbf{u}|^{1-n/2}(r^{-1}|\mathbf{u}|)^{\varepsilon},$$

which is satisfactory for (4.20). Suppose now that $|\mathbf{u}| > K^{(n+2\varepsilon)/n}r^{-2\varepsilon/n}$ and write

$$T = c(r^{-1}|\mathbf{u}|)^{\varepsilon/2n},$$

for a suitable constant $c > 0$ depending only on n. Then we claim that $T \gg_S 1$ and $|\mathbf{u}| \geqslant KT^2$ if c is chosen to be large enough. To see the former inequality we note that $r^{-1}|\mathbf{u}| > K^{(n+2\varepsilon)/n}r^{-1-2\varepsilon/n} \gg_S 1$, since $r \ll_S K$. Moreover, the latter inequality holds if and only if $|\mathbf{u}| \geqslant c^{2n/(n-\varepsilon)}K^{n/(n-\varepsilon)}r^{-\varepsilon/(n-\varepsilon)}$. But this is easily seen to hold when $|\mathbf{u}| > K^{(n+2\varepsilon)/n}r^{-2\varepsilon/n}$ and c is chosen to be suitably large, since $r \ll_S K$. Hence we may apply (4.19) to deduce that

$$J_r(\mathbf{u};\omega,f) \ll_{N,S} KT^{-N} + \frac{K^{3n/2+1}A_1^n}{|\Delta_Q|}r|\mathbf{u}|^{1-n/2}(r^{-1}|\mathbf{u}|)^{\varepsilon},$$

in this case. On taking N to be sufficiently large in terms of ε, we therefore complete the proof of (4.20) for any $\mathbf{u} \in \mathbb{R}^n$. We now insert this into (4.6), and then into (4.3) and (4.4), with $r = A_1^{-1/2}B^{-1}q$ and $\mathbf{v} = A_1^{-1/2}\mathbf{c}$, in order to conclude the proof of Lemma 11. $\square$

We end this section by considering the integral $I_q(\mathbf{0};w) = B^nI_r^*(\mathbf{0};w)$, for $r = A_1^{-1/2}B^{-1}q$ and any $w \in \mathcal{W}(S)$. A trivial application of Lemma 9, together

with (4.6), therefore yields $\partial^k I_r^*(\mathbf{0};w)/\partial r^k \ll_S r^{-k}$ for $k \in \{0,1\}$. Hence we may conclude from (4.3) and (4.4) that

$$\frac{\partial^k I_q(\mathbf{0};w)}{\partial q^k} \ll_S q^{-k} B^n, \tag{4.21}$$

for $k \in \{0,1\}$. We can achieve a finer estimate in the case $k=0$. Since (4.7) holds on $\mathrm{supp}(w)$, we may proceed as in the proof of Lemma 9 to conclude that

$$I_r^*(\mathbf{0};w) = \int_{-\infty}^{\infty} I(y)h(r,y)\mathrm{d}y,$$

where $I \in \mathcal{W}_1(S)$ is given by (4.8). Thus [6, Lemma 9] yields

$$I_r^*(\mathbf{0};w) = I(0) + O_{N,S}(r^N),$$

for any $N \geqslant 1$. In fact [6, Lemma 9] is stated under the assumption that $r \leqslant 1$. It is easy to see, however, that the result holds trivially if $r \geqslant 1$, since then $I(0) \ll_S 1 \leqslant r^N$. The integral $I(0)$ is a non-negative constant that is related to the singular integral, whose value depends only upon the weight w and the quadratic form Q. While the precise value of $I(0) = I(0;w)$ is unimportant for our purposes we will need the following upper bound.

Lemma 12. *Let $w \in \mathcal{W}(S)$. Then we have $I(0;w) \ll_S A_1^{1/2}\|Q\|^{-1/2}$.*

Proof. On relabelling the coefficients of Q, we may assume without loss of generality that

$$Q(\mathbf{x}) = A_1x_1^2 - \sigma_2A_2x_2^2 - \cdots - \sigma_nA_nx_n^2,$$

with $A_1,\dots,A_n > 0$ and $\sigma_2,\dots,\sigma_n \in \{-1,+1\}$. It therefore follows from a simple change of variables that

$$I(0;w) \leqslant A_1 \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{w(A_1^{-1/2}P(\mathbf{y}), A_2^{-1/2}x_2,\dots,A_n^{-1/2}x_n)\mathrm{d}\mathbf{y}}{|\Delta_Q|^{1/2}P(\mathbf{y})},$$

where the integral is over all $\mathbf{y} = (x_2,\dots,x_n) \in \mathbb{R}^{n-1}$, and

$$P(\mathbf{y}) = \sqrt{\sigma_2x_2^2 + \cdots + \sigma_nx_n^2}.$$

But the integrand here vanishes unless $A_1^{1/2} \ll_S P(\mathbf{y}) \ll_S A_1^{1/2}$. Hence we have

$$I(0;w) \ll_S A_1^{1/2}|\Delta_Q|^{-1/2}\mathcal{V}(Q),$$

where $\mathcal{V}(Q)$ is the volume of $\mathbf{y} = (x_2,\dots,x_n) \in \mathbb{R}^{n-1}$ for which $P(\mathbf{y}) \ll_S A_1^{1/2}$ and $x_i \ll_S A_i^{1/2}$, for $2 \leqslant i \leqslant n$. In order to complete the proof of Lemma 12 it will therefore suffice to show that $\mathcal{V}(Q) \ll_S |\Delta_Q|^{1/2}\|Q\|^{-1/2}$,

This inequality is trivial if $A_1 = \|Q\|$, since $\mathcal{V}(Q) \ll_S |A_2\cdots A_n|^{1/2}$. Alternatively, on supposing without loss of generality that $\|Q\| = A_n$, we fix values of $x_2,\dots,x_{n-1}$ and estimate the volume of $x_n \in \mathbb{R}$ such that $x_n \ll_S A_n^{1/2}$ and $|x_n^2 - \delta| \ll_S A_1$, where $\delta = \delta(x_2,\dots,x_{n-1})$. The latter inequality implies that this volume is $O_S(A_1^{1/2})$, which therefore leads to the overall estimate

$$\mathcal{V}(Q) \ll_S |A_1\cdots A_{n-1}|^{1/2} = |\Delta_Q|^{1/2}\|Q\|^{-1/2},$$

as required. □

Write $I(0;w^\dagger) = A_1\sigma_\infty(Q)$, where $w^\dagger \in \mathcal{W}(n)$ is given by (2.5). Then on combining Lemma 12 with our arguments above, we have therefore established the following result.

Lemma 13. *There exists a non-negative constant $\sigma_\infty(Q)$ such that*

$$I_q(\mathbf{0}) = A_1\sigma_\infty(Q)B^n + O_{n,N}\big(A_1^{-N/2}q^N B^{n-N}\big),$$

for any $N \geqslant 1$, with

$$\sigma_\infty(Q) \ll_n A_1^{-1/2}\|Q\|^{-1/2}.$$

5. Derivation of Proposition 1

In this section we are going to derive Proposition 1. Let $n \geqslant 5$ and recall the choice $X = A_1^{1/2}B$ that was made in (2.10). It therefore follows from (2.9) that

$$N_{w^\dagger}(Q;B) = \frac{c_B}{A_1B^2}\sum_{\mathbf{c}\in\mathbb{Z}^n}\sum_{q=1}^{\infty} q^{-n}S_q(\mathbf{c})I_q(\mathbf{c}), \tag{5.1}$$

where $c_B = 1 + O_N(B^{-N})$ for any $N \geqslant 1$, $S_q(\mathbf{c})$ is given by (2.7), and $I_q(\mathbf{c})$ is given by (2.8). Let $\varepsilon > 0$ and $P \geqslant 1$. Then it follows from Lemma 2, together with Lemma 10 and the fact that $I_q(\mathbf{c}) = 0$ for $q \gg_n B\|Q\|/A_1^{1/2}$, that the contribution to the right hand side of (5.1) from $|\mathbf{c}| > P$ is

$$\begin{aligned}
&\ll_{n,N} \frac{\|Q\|}{A_1^{3/2}}B^{n-1}\sum_{q\ll_n B\|Q\|/A_1^{1/2}} q^{-n-1}|S_q(\mathbf{c})|\sum_{|\mathbf{c}|>P}\frac{\|Q\|^N}{A_1^{N/2}|\mathbf{c}|^N}\\
&\ll_{\varepsilon,n,N} \frac{\|Q\|^{1+\varepsilon}|\Delta_Q|^{1/2}}{A_1^{3/2}}B^{n-1+\varepsilon}\frac{\|Q\|^N}{A_1^{N/2}P^{N-n}},
\end{aligned}$$

for any $N > n$. But this is clearly

$$\ll_{\varepsilon,n,M} \frac{\|Q\|^{n+1+\varepsilon}|\Delta_Q|^{1/2}}{A_1^{n/2+3/2}}B^{n-1+\varepsilon}\frac{\|Q\|^M}{A_1^{M/2}P^M}, \tag{5.2}$$

for any $M \geqslant 1$. Turning to the contribution from $1 \leqslant |\mathbf{c}| \leqslant P$, we employ Lemma 11 to deduce that

$$I_q(\mathbf{c}) \ll_{\varepsilon,n} \frac{\|Q\|^{3n/2+1+\varepsilon}}{A_1^{n/2+1}|\Delta_Q|}B^{n/2+1+\varepsilon}q^{n/2-1}|\mathbf{c}|^{1-n/2+\varepsilon}.$$

On combining this with Lemma 2, we therefore obtain

$$\begin{aligned}
\sum_{q=1}^{\infty} q^{-n}S_q(\mathbf{c})I_q(\mathbf{c}) &\ll \max_{Y\ll_n B\|Q\|/A_1^{1/2}}\sum_{j\leqslant \log Y}\sum_{2^{j-1}<q\leqslant 2^j} q^{-n}|S_q(\mathbf{c})I_q(\mathbf{c})|\\
&\ll_{\varepsilon,n} \frac{\|Q\|^{3n/2+1+\varepsilon}}{A_1^{n/2+1}|\Delta_Q|^{1/2}}B^{n/2+1+\varepsilon}\Big(\frac{B\|Q\|}{A_1^{1/2}}\Big)^{(1+\delta_n)/2}|\mathbf{c}|^{1-n/2+\varepsilon}\\
&\ll_{\varepsilon,n} \frac{\|Q\|^{3n/2+2+\varepsilon}}{A_1^{n/2+3/2}|\Delta_Q|^{1/2}}B^{(n+3+\delta_n)/2+\varepsilon}|\mathbf{c}|^{1-n/2+\varepsilon},
\end{aligned}$$

when $1 \leqslant |\mathbf{c}| \leqslant P$. Summing over such values of $\mathbf{c}$ we therefore deduce that the contribution to the right hand side of (5.1) from $1 \leqslant |\mathbf{c}| \leqslant P$ is

$$\ll_{\varepsilon,n} \frac{\|Q\|^{3n/2+2+\varepsilon}}{A_1^{n/2+5/2}|\Delta_Q|^{1/2}} B^{(n-1+\delta_n)/2+\varepsilon} P^{n/2+1+\varepsilon}.$$

Once combined with (5.2), we see that the overall contribution from $\mathbf{c} \neq \mathbf{0}$ is

$$\ll_{\varepsilon,n,M} \frac{\|Q\|^{n+1+\varepsilon}|\Delta_Q|^{1/2}}{A_1^{n/2+3/2}} B^{(n-1+\delta_n)/2+\varepsilon}\Big(\frac{\|Q\|^M B^{(n-1)/2}}{A_1^{M/2}P^M} + \frac{\|Q\|^{n/2+1}P^{n/2+1+\varepsilon}}{A_1|\Delta_Q|}\Big),$$

for any $M \geqslant 1$. Taking $M = \lceil (n-1)/(2\varepsilon)\rceil$ and $P = |\Delta_Q|^{1/M}\|Q\|B^\varepsilon/A_1^{1/2}$, we therefore see that there is a contribution of

$$\begin{aligned}
&\ll_{\varepsilon,n} \frac{\|Q\|^{n+1+\varepsilon}|\Delta_Q|^{1/2}}{A_1^{n/2+3/2}} B^{(n-1+\delta_n)/2+\varepsilon}\Big(\frac{1}{|\Delta_Q|} + \frac{\|Q\|^{n+2}}{A_1^{n/4+3/2}|\Delta_Q|}\Big)\\
&\ll_{\varepsilon,n} \frac{\|Q\|^{2n+3+\varepsilon}}{A_1^{3n/4+3}|\Delta_Q|^{1/2}} B^{(n-1+\delta_n)/2+\varepsilon}
\end{aligned} \tag{5.3}$$

to the right hand side of (5.1) from those $\mathbf{c} \neq \mathbf{0}$.

It remains to handle the contribution from the case $\mathbf{c} = \mathbf{0}$. For this it follows from Lemma 2 and (4.21) that for any $Y \geqslant 1$, we have

$$\sum_{Y/2<q\leqslant Y} q^{-n}S_q(\mathbf{0})I_q(\mathbf{0}) \ll_{\varepsilon,n} |\Delta_Q|^{1/2+\varepsilon}B^nY^{(3+\delta_n-n)/2+\varepsilon}.$$

On summing over dyadic intervals for Y such that $B \leqslant Y \ll_n B\|Q\|/A_1^{1/2}$, we deduce that the overall contribution to the right hand side of (5.1) from $\mathbf{c} = \mathbf{0}$ and $q \geqslant B$, is

$$\ll_{\varepsilon,n} A_1^{-1}|\Delta_Q|^{1/2+\varepsilon}B^{(n-1+\delta_n)/2+\varepsilon}. \tag{5.4}$$

Finally we note that an application of (3.2), together with Lemma 13, reveals that the contribution from $\mathbf{c} = \mathbf{0}$ and $q \leqslant B$ is

$$\begin{aligned}
\frac{c_B}{A_1B^2}\sum_{q\leqslant B} q^{-n}S_q(\mathbf{0})I_q(\mathbf{0}) =&\sigma_\infty(Q)\mathfrak{S}(Q)B^{n-2}\\
&+ O_{\varepsilon,n}(A_1^{-1}|\Delta_Q|^{1/2+\varepsilon}B^{(n-1+\delta_n)/2+\varepsilon})\\
&+ O_{n,N}\Big(\frac{B^{n-2-N}}{A_1^{N/2+1}}\sum_{q\leqslant B} q^{N-n}|S_q(\mathbf{0})|\Big),
\end{aligned}$$

for any $N \geqslant 1$. On selecting $N = (n-3-\delta_n)/2$, and applying Lemma 2, we deduce that the error terms in this estimate are also bounded by (5.4). Observe that

$$\frac{|\Delta_Q|^{1/2}}{A_1} = \frac{|\Delta_Q|}{A_1|\Delta_Q|^{1/2}} \leqslant \frac{\|Q\|^n}{A_1|\Delta_Q|^{1/2}} \leqslant \frac{\|Q\|^{2n+3}}{A_1^{3n/4+3}|\Delta_Q|^{1/2}}.$$

We may now combine these inequalities with (5.3) and (5.4) in (5.1), in order to complete the proof of Proposition 1.

6. Derivation of Proposition 2

We proceed as in the previous section, much of which carries over to this setting. Let $n=4$ and assume that Δ_Q is not a square. Then the above argument suffices to handle the terms with $|\mathbf{c}|\geqslant P$, or with $1\leqslant|\mathbf{c}|\leqslant P$ and $Q^{-1}(\mathbf{c})\neq 0$, where $P=|\Delta_Q|^\varepsilon\|Q\|B^\varepsilon/A_1^{1/2}$. Hence it follows that

$$N_{w^\dagger}(Q;B)=\frac{c_B}{A_1B^2}\sum_{\substack{\mathbf{c}\in\mathbb{Z}^4\\ Q^{-1}(\mathbf{c})=0\\ |\mathbf{c}|\leqslant P}}\sum_{q\ll B\|Q\|/A_1^{1/2}}q^{-4}S_q(\mathbf{c})I_q(\mathbf{c})+O_\varepsilon\Big(\frac{\|Q\|^{11+\varepsilon}}{A_1^6|\Delta_Q|^{1/2}}B^{3/2+\varepsilon}\Big),$$

where $c_B=1+O_N(B^{-N})$. When $1\leqslant|\mathbf{c}|\leqslant P$ and $Q^{-1}(\mathbf{c})=0$ we may use partial summation, based on Lemmas 8 and 11. Now the latter result implies that

$$\frac{\partial^k I_q(\mathbf{c})}{\partial q^k}\ll_\varepsilon\frac{\|Q\|^{7+\varepsilon}}{A_1^3|\Delta_Q|}B^{3+\varepsilon}q^{1-k}|\mathbf{c}|^{-1+\varepsilon},$$

for $k\in\{0,1\}$. Once combined with the former, we see that

$$\sum_{Y/2<q\leqslant Y}q^{-4}S_q(\mathbf{c})I_q(\mathbf{c})\ll_\varepsilon\frac{\|Q\|^{7+\varepsilon}}{A_1^3|\Delta_Q|^{1/4}}\frac{B^{3+\varepsilon}Y^{1/2+\varepsilon}}{|\mathbf{c}|^{1-\varepsilon}}\ll_\varepsilon\frac{\|Q\|^{15/2+\varepsilon}}{A_1^{13/4}|\Delta_Q|^{1/4}}\frac{B^{7/2+\varepsilon}}{|\mathbf{c}|^{1-\varepsilon}},$$

for any $Y\ll_\varepsilon B\|Q\|/A_1^{1/2}$. Now it follows from (1.6) that there are at most $O_{\varepsilon,n}(P^{2+\varepsilon})$ vectors $\mathbf{c}\in\mathbb{Z}^4$ for which $|\mathbf{c}|\leqslant P$ and $Q^{-1}(\mathbf{c})=0$. Hence we conclude that

$$N_{w^\dagger}(Q;B)=\frac{1}{A_1B^2}\sum_{q\ll B\|Q\|/A_1^{1/2}}q^{-4}S_q(\mathbf{0})I_q(\mathbf{0})+O_\varepsilon\Big(\frac{\|Q\|^{11+\varepsilon}}{A_1^6|\Delta_Q|^{1/2}}B^{3/2+\varepsilon}\Big).$$

To handle the contribution from large q we employ partial summation again, this time based on Lemma 8 and (4.21). Thus we obtain

$$\sum_{Y/2<q\leqslant Y}q^{-4}S_q(\mathbf{0})I_q(\mathbf{0})\ll_\varepsilon|\Delta_Q|^{3/4+\varepsilon}B^4Y^{-1/2+\varepsilon}.$$

for any $Y\geqslant 1$. On summing over dyadic intervals for Y in the interval $B^{1-\varepsilon}\leqslant Y\ll B\|Q\|/A_1^{1/2}$, we easily deduce that terms with $\mathbf{c}=\mathbf{0}$ and $q\geqslant B^{1-\varepsilon}$ contribute $O_\varepsilon(A_1^{-1}|\Delta_Q|^{3/4+\varepsilon}B^{3/2+\varepsilon})$ to $N_{w^\dagger}(Q;B)$, which is satisfactory. Finally, for $q\leqslant B^{1-\varepsilon}$ we may apply Lemma 13 and (3.10), together with the second part of Lemma 6, in order to conclude that

$$N_{w^\dagger}(Q;B)=\sigma_\infty(Q)\mathfrak{S}(Q)B^2+O_\varepsilon\Big(\frac{\|Q\|^{11+\varepsilon}}{A_1^6|\Delta_Q|^{1/2}}B^{3/2+\varepsilon}\Big).$$

This completes the proof of Proposition 2.

7. The singular series

In this section we establish Proposition 3. Let $n\geqslant 4$ and assume that Δ_Q is not a square when $n=4$. Then (3.1) holds, and we have

$$\mathfrak{S}(Q)=\prod_p D_p(n;\mathbf{0}),$$

in the notation of (3.4). We begin by handling the factors $D_p(n;\mathbf{0})$, for which $p\nmid \Delta_Q$. Suppose first that $p=2$. Then an application of [2, Lemma 4] reveals that

$$D_2(n;\mathbf{0})=1+O_{\varepsilon,n}\Big(\sum_{k\geqslant 2}2^{k(1-n/2+\varepsilon)}\Big)\ll_n 1,$$

since $n\geqslant 4$. Suppose now that $p>2$ and write $\chi_Q(p)=(\frac{\Delta_Q}{p})$, as usual. We may therefore combine (3.6) with the proof of [2, Eqn. (5.2)], in order to deduce that

$$D_p(n;\mathbf{0})=\begin{cases}(1-\chi_Q(p)p^{-1})^{-1}(1+O(p^{-3/2})), & \text{if } n=4,\\ 1+O_{\varepsilon,n}(p^{-3/2+\varepsilon}), & \text{if } n\geqslant 5.\end{cases}$$

Now Lemma 7 implies that $L(1,\chi_Q)\ll_\varepsilon |\Delta_Q|^\varepsilon$. Hence

$$D_2(n;\mathbf{0})\prod_{p\nmid 2\Delta_Q}D_p(n;\mathbf{0})\ll_{\varepsilon,n}\begin{cases}|\Delta_Q|^\varepsilon, & \text{if } n=4 \text{ and } \Delta_Q \text{ not square},\\ 1, & \text{if } n\geqslant 5.\end{cases}\tag{7.1}$$

We now turn to an upper bound for the factors $\sigma_p=D_p(n;\mathbf{0})$, for odd $p\mid\Delta_Q$. Recall that

$$\sigma_p=\lim_{k\to\infty}p^{-k(n-1)}N_k(p),\quad N_k(p)=\#\{\mathbf{x}\ (\mathrm{mod}\ p^k): Q(\mathbf{x})\equiv 0\ (\mathrm{mod}\ p^k)\}.$$

After relabelling the indices we may assume that there exists $d\in\mathbb{N}$ such that

$$(A_1,\dots,A_n)=(a_1,\dots,a_r,p^db_1,\dots,p^db_s),$$

with $p\nmid a_1\cdots a_r$. Moreover, we may suppose that $r+s=n$ and there is an index $1\leqslant i\leqslant s$ such that $p\nmid b_i$. We have $r,s\geqslant 1$, since $p\mid\Delta_Q$ and the highest common factor of $A_1,\dots,A_n$ is assumed to be 1. Observe that for any fixed integers a,b such that $p\nmid a$, and any $k\in\mathbb{N}$, the number of positive integers $n\leqslant p^k$ such that $an^2\equiv b\,(\mathrm{mod}\ p^k)$ is at most 2. We proceed to show that

$$N_k(p)\leqslant 4p^{k(n-1)},\tag{7.2}$$

for any $k\geqslant 1$. If $k\leqslant d$ then it easily follows that

$$N_k(p)=p^{ks}\#\{x_1,\dots,x_r\ (\mathrm{mod}\ p^k):\textstyle\sum_{i=1}^r a_ix_i^2\equiv 0\ (\mathrm{mod}\ p^k)\}\leqslant 2p^{k(n-1)},$$

which is satisfactory for (7.2). Assume now that $k>d$. On writing $\mathbf{x}=(\mathbf{y},\mathbf{z})$ for $\mathbf{y}=(y_1,\dots,y_r)$ and $\mathbf{z}=(z_1,\dots,z_s)$ modulo p^k, we see that

$$\begin{aligned}N_k(p)&=\#\{\mathbf{y},\mathbf{z}\ (\mathrm{mod}\ p^k):\textstyle\sum_{i=1}^r a_iy_i^2+p^d\sum_{i=1}^s b_iz_i^2\equiv 0\ (\mathrm{mod}\ p^k)\}\\ &=\sum_{\substack{\mathbf{y}\,(\mathrm{mod}\ p^k)\\ \exists w\in\mathbb{Z}:\ \sum_{i=1}^r a_iy_i^2=p^dw}}\#\{\mathbf{z}\ (\mathrm{mod}\ p^k):\textstyle\sum_{i=1}^s b_iz_i^2\equiv w\ (\mathrm{mod}\ p^{k-d})\}.\end{aligned}$$

Now the summand here is plainly equal to

$$p^{ds}\#\{\mathbf{z}\ (\mathrm{mod}\ p^{k-d}):\textstyle\sum_{i=1}^s b_iz_i^2\equiv w\ (\mathrm{mod}\ p^{k-d})\}\leqslant 2p^{ds+(k-d)(s-1)},$$

since there exists at least one value of $b_1,\dots,b_s$ that is not divisible by p. But then it follows that for $k>d$ we have

$$\begin{aligned}N_k(p)&\leqslant 2p^{k(s-1)+d}\#\{\mathbf{y}\ (\mathrm{mod}\ p^k):\textstyle\sum_{i=1}^r a_iy_i^2\equiv 0\ (\mathrm{mod}\ p^d)\}\\ &\leqslant 4p^{k(s-1)+d}p^{(k-d)r+d(r-1)}=4p^{k(n-1)}.\end{aligned}$$

This too is satisfactory for (7.2). We have therefore shown that $\sigma_p \leqslant 4$ when $n \geqslant 4$ and $p \mid \Delta_Q$ is an odd prime. Once combined with (7.1), this shows that

$$\mathfrak{S}(Q) \ll_{\varepsilon,n} |\Delta_Q|^\varepsilon \prod_{p|\Delta_Q} 4 \ll_{\varepsilon,n} |\Delta_Q|^\varepsilon,$$

and so completes the proof of Proposition 3.

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